Readers' Forum

Brief discussion of previous investigations in the aerospace sciences and technical comments on papers published in the AIAA Journal *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

Comment on "Analytical Solutions for the *n*th Derivatives of Eigenvalues and Eigenvectors for a Nonlinear Eigenvalue Problem"

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A RECENT synoptic of Jankovic[1] announced six theorems concerning derivatives with respect to a scalar parameter π of the eigenvalues λ and normalized eigenvectors x of the nonlinear eigenvalue problem

$$A(\pi,\lambda)x = 0 \tag{1}$$

with the normalizing condition

$$x^{\dagger}K(\pi,\lambda)x = 1 \tag{2}$$

Here $A(\pi,\lambda)$ and $K(\pi,\lambda)$ are $n \times n$ matrices and $x^{\dagger}$ is the complex conjugate transpose of x. However, Example 1, given later in this Comment, shows that his first four theorems, and his Eq. (13) from which they are deduced, are not true in general, even in the linear case.

Assumptions stated in Ref. 1 are (i) λ is an unrepeated eigenvalue, (ii) $K(\pi,\lambda)$ is positive definite hermitian, (iii) π is real-valued, and (iv) all required derivatives exist. Let the subscripts π and λ denote partial derivatives with respect to π and λ respectively, and let the superscript (n) denote the nth total derivative with respect to π. Define $\alpha = x^{\dagger}K_{\lambda}x$ and $\beta = x^{\dagger}K_{\pi}x$. The first four theorems in Ref. 1 can now be stated as follows:

"If $\alpha = 0$ then

$$-(A_{\lambda}^{-1}A_{\pi})^{(n)}x = \lambda^{(n+1)}x, \; n = 0,1,\ldots \tag{3}$$

$$\lambda^{(n)} = -x^{\dagger}K(A_{\lambda}^{-1}A_{\pi})^{(n-1)}x, \; n = 1,2,\ldots \tag{4}$$

$$x^{(n)} = -(\beta x)^{(n-1)}/2, \; n = 1,2,\ldots \tag{5}$$

and

$$A^{(n-k)}x^{(k)} = 0, \; k = 0,1,\ldots,n, \; n = 1,2,\ldots\text{"} \tag{6}$$

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Example 1: Let $K(\pi,\lambda) = I$ (the identity) and

$$A(\pi,\lambda) = \begin{bmatrix} 1-\lambda & \pi \\ 1 & \pi-\lambda \end{bmatrix}$$

In this case $\alpha = \beta = 0$ and, when $\pi \neq -1$, assumptions (i) – (iv) are satisfied. For all π, $\lambda = 0$ is an eigenvalue with corresponding eigenvector $x = (1+\pi^2)^{-\frac{1}{2}}(\pi, -1)^{\dagger}$, so that $\lambda^{(1)} = 0$. For this eigenvalue, when $n = 0$, the left hand side (LHS) of Eq. (3) is $-(1+\pi^2)^{-\frac{1}{2}}(1,1)^{\dagger} \neq 0$ = right hand side (RHS). When $n = 1$, the LHS of Eq. (4) is $0 \neq (1-\pi)/(1+\pi^2)$ (in general) = RHS, while the LHS of Eq. (5) is $(1+\pi^2)^{-3/2}(1,\pi)^{\dagger} \neq 0$ = RHS. When $n = 1$ and $k = 0$ the LHS of Eq. (6) is $-(1+\pi^2)^{-\frac{1}{2}}(1,1)^{\dagger} \neq 0$ = RHS. The proofs in Jankovic's full paper are by induction and require Eqs. (3)-(6) to be true for these low values of n.

If $(PAx)_{\pi}$ in Jankovic's Theorem VI is replaced by $(PA_{\pi}x)$, his Theorems V and VI, which give results when $\alpha \neq 0$, appear to be true whenever his Eq. (7) is true, i.e. whenever $x^{\dagger}Kx_{\Pi}$ is real. However, as is already known,[2] (total) differentiation of Eq. (1) with respect to π and premultiplication by the corresponding left eigenvector, $y^{\dagger}$, of $A(\pi,\lambda)$ shows that

$$\lambda^{(1)} = -y^{\dagger}A_{\pi}x/y^{\dagger}A_{\lambda}x \tag{7}$$

provided only that $y^{\dagger}A_{\lambda}x \neq 0$. This generalization of the classical result for the linear case[3,4] is often simpler to use than Theorem V of Ref. 1 and does not require $\alpha \neq 0$ or $x^{\dagger}Kx_{\Pi}$ real. (This equation has been derived independently by Jankovic in a somewhat different manner in an unpublished work.[5]) In Eq. (4) of Ref. 1, A_{Π} should be A_{π}. Computation of higher derivatives of λ by differentiating Eq. (7) requires derivatives of eigenvectors.

The problem addressed by Jankovic is important in applications. Recent work on the linear case has produced analytical solutions[6-9], numerical techniques[10,11] and background theory,[12] but much work remains.

Acknowledgment

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Reply by the Author to A. L. Andrew and K.-W. E. Chu

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IN my original published paper[1] there were several obvious typographical mistakes that the authors of the Technical Comment have correctly identified. These are as follows:

A_Π in Eq. (4) should be replaced by A_π, which is obvious from Eq. (3) and (5); $(P\ \mathrm{Ax})_\pi$ in Theorem VI should be replaced by $(P\,Ax)_\pi$ in Theorem VI should be replaced by $(P\,A_\pi x)$ as can be seen from Eq. (15). The definition of the matrix P is missing in the paper and is given further below.

It is true that the Theorems I to IV are not valid in general for $\alpha = 0$ and their general validity was nowhere stated. They were derived under the assumption that $x_\Pi = -\beta x/2$ [Eq. (13)] which is not true in general. But, regardless of the value of β, Theorems I, II, III and IV are valid if the above assumption is valid; that is if the eigenvector derivative x_Π is proportional to the eigenvector itself in which case $Ax_\Pi = 0$ ($A_\Pi x = 0$). Therefore, for the example given by the authors, Theorems I to IV do apply for the second eigenvalue $\lambda = \pi + 1$ only. The authors failed to mention this. Thus, from Theorem II

$$\begin{aligned}\lambda_\Pi &= x^T A_\pi x \\ &= (x_1 x_2)\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \\ &= x_2^2\left(\frac{x_1}{x_2}+1\right) \\ &= \frac{1}{(\lambda-\pi)^2+1}(\lambda-\pi+1)\end{aligned}$$

and finally $\lambda_\Pi = 1$. Also from Theorem I, $x_\Pi = 0$.

However, Theorems V and VI for $\alpha \neq 0$ not only appear to be true when $x\dagger Kx_\Pi$ is real, as stated by Andrew and Chu, but are indeed true when $x\dagger Kx_\Pi$ is complex also. Since Andrew and Chu did not include a counter example to disprove Theorems V and VI, I am including a nonlinear example 1), which supports their validity, for a complex matrix A with complex eigenvalues and eigenvectors and complex $x\dagger Kx_\Pi$.

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First let me restate Theorems V and VI for completeness.

Theorem V: For a nonlinear eigenvalue problem previously defined, and for $\alpha \neq 0$, the nth derivative of the eigenvalue $\lambda^{(n)}$ with respect to the parameter π is

$$\lambda^{(n)} = 2\left[x\dagger K\left(\frac{PA_\pi}{\alpha}\right)x\right]^{(n-1)}$$

Theorem VI: For a nonlinear eigenvalue problem previously defined, and for $\alpha \neq 0$, the nth derivative of the eigenvector $x^{(n)}$ with respect to the parameter π is

$$x^{(n)} = -(P\,A_\pi x)^{(n-1)} - \frac{(\beta x)^{(n-1)}}{2}$$

where

$$P = (A - \tfrac{2}{\alpha}A_\lambda x x\dagger K)^{-1}$$

is assumed to exist which is for $A_\lambda \neq 0$.

Example 1:

Let

$$A = \begin{bmatrix} c\lambda^2 - \pi\lambda & b \\ a & 1\end{bmatrix}$$

where

λ = complex eigenvalue
π = real parameter
a, b, c = complex constants,

and

$$K = \begin{bmatrix} k & \lambda \\ \lambda^* & k\end{bmatrix}$$

= arbitrary Hermitian matrix

where

k = real constants,
λ^* = complex conjugate of λ.

The problem is to find the derivatives of eigenvalue λ_Π and eigenvector x_Π for a nonlinear eigenvalue problem

$$Ax = 0$$
$$x\dagger Kx = 1$$

From

$$\det A = c\lambda^2 - \pi\lambda - ab = 0$$

we find by implicit differentiation

$$\lambda_\Pi = \frac{\lambda}{2c\lambda - \pi}$$

and for the eigenvector x

$$x = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = x_1\begin{bmatrix}1\\ -a\end{bmatrix},\quad x_1 = \frac{1}{[k + a^*(ka - \lambda^*) - \lambda a]^{1/2}}$$

the derivatives, calculated by direct differentiation, are

$$x_{1\Pi} = \tfrac{1}{2}x_1^3(a\lambda_\Pi + a^*\lambda_\Pi^*);\ x_{2\Pi} = -\tfrac{1}{2}x_1^3(a^2\lambda_\Pi + |a|^2\lambda_\Pi^*).$$